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LETTER TO THE EDITOR

Diffusion and drift on percolation networks in an external field

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Abstract. The diffusion of a particle on a percolation network is studied in the presence of an external field which makes the particle more likely to move along the field than otherwise. It is shown that while in weak fields, less than a critical value, the mean displacement is linear in time; in the presence of strong fields, the particle has zero drift velocity but the mean displacement varies as t^{α} for large times t, where α is a non-universal field-dependent exponent less than one.

Diffusion on percolation networks (the 'ant in the labyrinth' problem) has been studied for several years (de Gennes 1976). It is known that systems close to the percolation threshold exhibit an anomalous diffusion regime in which the mean square displacement of the particle varies as a non-integral (<1) power of time (Gefen *et al* 1983) for displacements less than the percolation correlation length. In this letter, we argue that a similar non-classical diffusion and drift should be observable in the case of diffusion on percolation networks in the presence of strong fields, but in the limit of large times. The situation is physically realised approximately in the case of diffusion of ions in gels in chromatographic columns, or in the case of hopping electron conduction in doped semiconductors (Bottger and Bryskin 1980, Van lien and Shklovskii 1981, Van der Mere *et al* 1982) in the presence of a strong electric field; and our results may help in understanding them better.

Barma and Dhar (1983) studied the long-time drift behaviour of a particle performing a random walk on a percolation network in an external field, and argued that the drift velocity is a non-monotonic function of the field strength, and becomes zero at a critical (finite) value of the field strength. They claimed that for stronger fields the drift velocity remains zero, as the particle tends to become trapped near dead-end regions of the network. Pandey (1983) studied the problem using Monte Carlo simulations. He found that the mean square displacement of the diffusing particle keeps on increasing with time, and concluded that for any large but finite field strength, the drift velocity would be non-zero.

This letter attempts to resolve this controversy, and also to clarify the situation in the high-field regime where the drift velocity, defined as a large-time limit of the ratio of the mean displacement and time, is zero. In this case, it is shown that the particle shows a non-classical diffusion and drift, with the mean displacement varying as t^{α} for large times t, where α is a non-universal field-dependent exponent less than one.

We consider diffusion on a hypercubical lattice in d dimensions ($d \geqslant 2$) in which a finite fraction of sites have been removed. We study the motion of a particle starting

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at the origin and performing a random walk. $W_{mn}\,dt$ is the probability that the particle, when at site m, makes a transition to the site n in time dt. We assume that W_{mn} is zero unless m and n are nearest neighbours and are both occupied, it is W if n is a forward neighbour of m, and it is W/λ if n is a backward neighbour of m. λ is a field-dependent constant. We determine the long-time behaviour of $\langle R(t)\rangle$, the mean displacement at time t.

Below the percolation threshold, all clusters are finite and $\langle R(t)\rangle$ remains finite as $t \to \infty$. In the case of a fully occupied lattice, it is easy to show that $\langle R(t)\rangle$ is linear in t, and the proportionality constant, called the drift velocity, increases with the field.

Above the percolation threshold, in the presnce of disorder, there is a finite density of dead-end sites in the network. In strong fields, the escape probability from these sites is non-zero but small. The depth of a dead-end site from the backbone is measured in lattice units as the number of bonds against the field minus the number of bonds along the field in the shortest path connecting the dead-end site to the backbone of the percolation cluster. For a dead-end site of depth $d \gg 1$, the escape time varies as λ^d. The density of traps of depth d varies as $\exp(-d/\xi)$ for large d, where ξ is the percolation correlation length. (It is easy to derive upper and lower bounds on the density to show it must vary exponentially with d.) Hence the dead-end sites with escape time $\geqslant t$ have a density varying as $t^{-\alpha}$ where

$$\alpha = 1/(\xi \ln \lambda). \tag{1}$$

In strong fields, the particle spends most of its time in the dead-end branches of the network. At any large time t, it would be typically trapped or just emerging from dead-end branch with escape time $\geqslant t$. Thus, the mean distance travelled by the particle in time t is approximately equal to the mean free path before a dead-end trap with escape time $\geqslant t$ is encountered. Since the mean free path varies as the inverse of the density, we have for large t

$$\langle R(t)\rangle \sim t^{\alpha}, \qquad \text{for } \alpha < 1. \tag{2}$$

If $\alpha > 1$, then $\langle R(t)\rangle$ is linear in t. The trapping effect of dead-end sites is not strong enough, and the particle has a finite drift velocity. Note that α decreases to zero as λ is increased. The boundary between the drift and no-drift regimes is given by

$$\xi \ln \lambda = 1, \tag{3}$$

which is the condition derived by Barma and Dhar (1983) using a somewhat different argument.

The mean displacement of the particle perpendicular to the applied field is zero by symmetry. The mean square perpendicular displacement would also be expected to vary as t^{α} in strong fields. This is because the particle makes roughly t^{α} steps in time t if we eliminate re-traversals, and each of these steps has a random transverse (to the field) component.

These arguments given above are certainly not rigorous, but very plausible, and we would expect the conclusions to be valid in arbitrary dimensions. White and Barma (1984) have studied this problem more carefully in the special case of the Bethe lattice, and also the 'random comb' model (which models the random distribution of escape times of dead-end sites encountered along the path of the particle). They find the drift velocity going to zero for a finite value of the field strength on the Bethe lattice, and that $\langle R(t)\rangle$ varies as t^{α} in the random comb model, in agreement with the arguments presented here. The Monte Carlo results of Pandey also strongly suggest a power-law variation of mean square displacement with time. They are thus not in disagreement with the results of Barma and Dhar, but actually support them.

We have neglected the effect of traps along the backbone (in the terminology of Barma and Dhar) in the above analysis. It seems likely that the inclusion of these effects would not lead to a change in the value of the exponent in equation (2). Further research is necessary to establish this point conclusively.

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